

Perturbations of weakly expanding critical orbits

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Abstract

Let f be a polynomial or a rational function, which has r summable critical points. We prove that there exists an r -dimensional manifold Λ in an appropriate space containing f , such that, for every smooth curve in Λ through f , the ratio between parameter and dynamical derivatives along forward iterates of at least one summable point tends to a non-zero number.

1 Introduction

We say that a critical point c of a rational function f is *weakly expanding*, or *summable*, if, for the point $v = f(c)$ of the Riemann sphere $\bar{\mathbf{C}}$,

$$\sum_{n=0}^{\infty} \frac{1 + |f^n(v)|^2}{1 + |v|^2} \frac{1}{|(f^n)'(v)|} < \infty. \quad (1)$$

Throughout the paper, derivatives are standard derivatives of holomorphic maps; then the summand in (1) is a finite number for every $v \in \bar{\mathbf{C}}$ as soon as v is not a critical point of f^n .

Dynamical and statistical properties of one-dimensional and rational maps under different summability conditions have been investigated since [16], see e.g. [7], [3], [20], [22]. In the present paper, we study perturbations of polynomials and rational functions with several (possibly, not all) weakly expanding critical points. By its results and tools, the paper is a natural continuation of [11], [10].

Let us describe our main result in the particular case of a rational function f of degree $d \geq 2$ with all $2d - 2$ critical points simple. Assume r critical points c_1, \dots, c_r of f are summable, and the union $K = \cup_{j=1}^r \omega(c_j)$ of their ω -limit sets on the Riemann sphere satisfies a mild condition (see Section 3.3); for example, it is enough that K has zero Lebesgue measure on the plane. Consider a small

enough neighborhood X of f in the space of all rational functions of degree d . We prove that there exists a $2d - 2$ - dimensional manifold Λ_f in X containing f , such that every $g \in X$ is conjugated by a Mobius transformation to some $\tilde{g} \in \Lambda_f$ and such that there exists an r -dimensional submanifold Λ of Λ_f containing f , with the following property: for every family of maps $f_t \in \Lambda$, if $f_t(z) = f(z) + tu(z) + O(|t|^2)$ as $t \rightarrow 0$ and $u \neq 0$, then, for at least one critical point c_j , $1 \leq j \leq r$, the limit

$$\lim_{m \rightarrow \infty} \frac{\frac{d}{dt}|_{t=0} f_t^m(c_j(t))}{(f^{m-1})'(f(c_j))} = \sum_{n=0}^{\infty} \frac{u(f^n(c_j))}{(f^n)'(f(c_j))}$$

exists and is a *non-zero* number. (Here $c_j(t)$ is a critical point of f_t so that $c_j(0) = c_j$.) In the case $f_t(z) = z^d + t$, the limit above is the *similarity factor* between dynamical and parameter planes, see [19].

Let us list some cases when the set K (the union of the ω -limit sets of summable critical points) has zero Lebesgue measure.

(1) If a critical point c satisfies the Misiurewicz condition, i.e., c lies in the Julia set J of f and $\omega(c)$ contains no critical points and parabolic cycles, then $\omega(c)$ has Lebesgue measure zero (by Mane's theorem, $\omega(c)$ is a hyperbolic set, and then the bounded distortion property applies). Therefore, if c_1, \dots, c_r satisfy the Misiurewicz condition, the measure $|K|$ of K is zero, and the above result applies. In particular, $\frac{d}{dt}|_{t=0} f_t^m(c_j(t)) \rightarrow \infty$ as $m \rightarrow \infty$. On the other hand, the set $K = \cup_{i=1}^r \omega(c_i)$ is hyperbolic for f and, hence, is included in a holomorphic motion K_t for the family f_t . In particular, the speed $a'(0)$ of the points $a(t)$ of the motion is uniformly bounded. As a corollary, a high iterate $f_t^m(c_j(t))$ moves with the speed at $t = 0$, which is bigger than the speed of the point $a(t)$, where $a(0) = f^m(c_j)$. If $r = 2d - 2$ (i.e., all the critical points are simple and satisfy the Misiurewicz condition), this recovers a transversality result from [23] (see also [6] for a weak transversality result under the assumption that all critical points in the Julia set satisfy the Misiurewicz condition).

(2) If all critical points satisfy the Collet-Eckmann condition (which clearly implies the summability), and the ω -limit set of each of them is not the whole sphere, then the measure of their union is zero [17]. See also [18] for a rigidity result for Collet-Eckmann holomorphic maps.

(3) If *all* critical points in the Julia set J of a rational function f are summable, f has no neutral cycles, and J is not the whole sphere, then the Lebesgue measure of J is equal to zero [3], [20]. In particular, $|K| = 0$. Moreover, as in the case (2), if $J = \bar{\mathbf{C}}$, it is enough to assume that the ω -limit set of each of them is not the whole sphere, and then again $|K| = 0$ (the proof follows from [22], as explained in [21]).

Main results of the paper are contained in Theorem 1 (+ Comment 1) for polynomials, and in Theorem 2 and Corollary 3.1 for rational functions. See Comment 4 for a generalization, which takes into account also non-repelling cycles.

As usual, the polynomial case is more transparent and technically easier, so we start with this case. In the course of the proof, we clarify the meaning of coefficients of some formulas related to a Ruelle operator that have been known since [9], [12], see (5), (66).

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2 Polynomials

2.1 Polynomial spaces

Let f be a monic centered polynomial of degree $d \geq 2$, and C the set of different critical points of f . Consider the space Π_d of all monic and centered polynomials of the same degree d . Vector of coefficients of $g \in \Pi_d$ defines a (global) coordinate in Π_d and identifies Π_d with \mathbf{C}^{d-1} . Now, we consider a local subspace $\Pi_{d,\bar{p}}$ of Π_d associated to the polynomial f , as in [10]. Namely, let the set C consist of p different points c_1, \dots, c_p , with the vector of multiplicities $\bar{p} = \{m_1, m_2, \dots, m_p\}$, so that $f'(z) = d \prod_{i=1}^p (z - c_i)^{m_i}$. Then the space $\Pi_{d,\bar{p}}$ is defined locally near f as the set of $g \in \Pi_d$ with the same number p of different critical points $c_1(g), \dots, c_p(g)$ and the same vector of multiplicities \bar{p} . Here $c_i(g)$ is close to $c_i = c_i(f)$ and has the multiplicity m_i , $1 \leq i \leq p$. Note that $\sum_{i=1}^p m_i c_i(g) = 0$ because g is centered.

In particular, $\Pi_{d,\bar{p}}$ consists of all monic centered polynomials of degree d close to f if and only if all the critical points of f are simple. At the other extreme case, the space $\Pi_{d,\bar{1}}$ consists of the unicritical family $z^d + v$.

Consider the vector of critical values $V(g) = \{v_1(g), \dots, v_p(g)\}$, where $v_i(g) = g(c_i(g))$. The following fact is proved in Proposition 1 of [10]: $\Pi_{d,\bar{p}}$ is a p -dimensional complex analytic manifold, and the vector $V(g)$ is a local analytic coordinate.

Throughout the paper we use the following notations. Let f be a polynomial (rational function) included in a subset N of polynomials (rational functions) g . Assume that N has the structure of a complex-analytic l -dimensional manifold with (local) coordinates $\bar{x}(g) = \{x_1(g), \dots, x_l(g)\}$ corresponding to $g \in N$. Denote $\bar{x} = \bar{x}(f)$. Now, if P is a (scalar) function which is defined and analytic in a neighborhood of f in the space N , we denote by $\frac{\partial P}{\partial x_i}$ the partial derivative of $P(g)$ w.r.t. $x_i(g)$ calculated at the point \bar{x} , i.e., at $g = f$. Furthermore, if $P = g^m$, we denote $\frac{\partial f^m}{\partial x_i}(z)$ to be $\frac{\partial g^m}{\partial x_i}(z)$. For a rational function $g(z)$, g' always means the derivative w.r.t. $z \in \mathbf{C}$. For instance, for $N = \Pi_{d,\bar{p}}$, $\frac{\partial g'}{\partial v_k}$ means $\frac{\partial(\partial g / \partial z)}{\partial v_k}$ calculated

at the point $V(f)$. Note also that for a critical point $c(g)$ of g considered as a function of $V(g)$, we have: $\frac{\partial(g^m(c(g)))}{\partial v_k} = \frac{\partial f^m}{\partial v_i}(c)$, where $c = c(f)$.

2.2 Main result

Theorem 1 (a) Let c be a weakly expanding critical point of f and $v = f(c)$. Then, for every $k = 1, \dots, p$, the following limit exists:

$$L(c, v_k) := \lim_{m \rightarrow \infty} \frac{\frac{\partial(f^m(c))}{\partial v_k}}{(f^{m-1})'(v)}. \quad (2)$$

(b) Suppose that c_1, \dots, c_r are pairwise different weakly expanding critical points of f . Then the rank of the matrix

$$\mathbf{L} = (L(c_j, v_k))_{1 \leq j \leq r, 1 \leq k \leq p} \quad (3)$$

is equal to r , i.e., maximal.

Comment 1 Part (b) has the following geometric re-formulation. Let $1 \leq k_1 < \dots < k_r \leq p$ be indexes, for which determinant of the square matrix $(L(c_j, v_{k_i}))_{1 \leq j \leq r, 1 \leq i \leq r}$ is non-zero. We define a local r -dimensional submanifold Λ as the set of all $g \in \Pi_{d, \bar{p}}$, for which $v_i(g) = 0$ for every $i \neq k_1, k_2, \dots, k_r$. Consider a family (curve) of maps $f_t \in \Lambda$ through f , such that $f_t(z) = f(z) + tu(z) + O(|t|^2)$ as $t \rightarrow 0$. If $u \neq 0$, then, for at least one weakly expanding critical point c_j of f ,

$$\lim_{m \rightarrow \infty} \frac{\frac{d}{dt}|_{t=0} f_t^m(c_j(t))}{(f^{m-1})'(v_j)} \neq 0. \quad (4)$$

Here $c_j(t)$ is a holomorphic function with $c_j(0) = c_j$, so that $c_j(t)$ is a critical point of f_t , and $v_j = f(c_j)$.

In particular, if all critical points of f are simple and weakly expanding, then (4) holds for every curve f_t in Π_d through f with a non-degenerate tangent vector at f .

Furthermore, if $f(0)$ and all the critical points of f are real, the above maps and spaces can be taken real.

Indeed, let $v_k(t)$ be the critical value of f_t , such that $v_k(0) = v_k$. Denote $a_k = v'_k(0)$. As $f_t \in \Lambda$, $u(z) = \sum_{i=1}^r a_{k_i} \frac{\partial f}{\partial v_{k_i}}(z)$, where at least one of a_{k_i} must be non zero. On the other hand, the limit R_j in (4) can be represented as $R_j = \sum_{k=1}^p a_k L(c_j, v_k)$. Therefore, $R_j = \sum_{i=1}^r a_{k_i} L(c_j, v_{k_i})$, where at least one of a_{k_i} is not zero. Now, if we assume that $R_j = 0$ for every $1 \leq j \leq r$, then the matrix $(L(c_j, v_{k_i}))_{1 \leq j \leq r, 1 \leq i \leq r}$ degenerates, a contradiction.

2.3 The Ruelle operator

Once the part (a) of Theorem 1 is verified, the proof of part (b) follows closely by the proof of Corollary 1(b) of [11] using Proposition 1 below. The main tool is the following linear operator $T = T_f$ acting on functions as follows:

$$T\psi(x) = \sum_{w:f(w)=x} \frac{\psi(w)}{(f'(w))^2},$$

provided x is not a critical value of f .

Proposition 1 *We have (in formal series):*

$$\varphi_{z,\lambda}(x) - \lambda(T\varphi_{z,\lambda})(x) = \frac{1}{z-x} + \lambda \sum_{k=1}^p \frac{1}{v_k - x} \Phi_k(\lambda, z), \quad (5)$$

where

$$\varphi_{z,\lambda}(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(f^n)'(z)} \frac{1}{f^n(z) - x}, \quad (6)$$

and

$$\Phi_k(\lambda, z) = \sum_{n=0}^{\infty} \lambda^{n+1} \frac{\frac{\partial f}{\partial v_k}(f^n(z))}{(f^{n+1})'(z)}, \quad (7)$$

for λ complex parameter and z, x complex variables.

We will apply this proposition for $z = v_j$ and $\lambda = 1$ (under the summability condition on c_j). Note that in the case of simple critical points, $\Phi_k(\lambda, z) = \varphi_{z,\lambda}(c_k)$, and with $\Phi_k(\lambda, z)$ in such form, (5) as well as (66) appear for the first time in [12] (where it is written in an operator form). The only new (and crucial) ingredient of Proposition 1 (as well as Proposition 9 of the next Section) is the representation of the coefficients $\Phi_k(\lambda, z)$ in (5) via derivatives w.r.t. the local coordinates in the appropriate space of maps.

2.4 Proof of Part (a) of Theorem 1

First, we calculate partial derivatives of a function $f \in \Pi_{d,\bar{p}}$ w.r.t. the local coordinates.

Proposition 2 *For every $k = 1, \dots, p$, the function $\frac{\partial f}{\partial v_k}(z)$ is a polynomial $p_k(z)$ of degree at most $d-2$, which is uniquely characterized by the following condition: $p_k(z) - 1$ has zero at c_k of order at least m_k , while for every $j \neq k$, $p_k(z)$ has zero at c_j of order at least m_j . In particular, $\frac{\partial f}{\partial v_k}(c_j) = \delta_{j,k}$ (here and later on we use the notation $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq k$); if c_k is simple (i.e., $m_k = 1$), then*

$$\frac{\partial f}{\partial v_k}(z) = \frac{f'(z)}{f''(c_k)(z - c_k)}. \quad (8)$$

Proof. Since the coefficients of $g \in \Pi_{d,\bar{p}}$ are holomorphic functions of $V(g)$ and g is centered, the function $\frac{\partial f}{\partial v_k}(z)$ is indeed a polynomial in z of degree at most $d - 2$. Hence, it is enough to check that it satisfies the characteristic property of the polynomial $p_k(z)$. The rest of the proof is a “proper” calculation. (It is quite general; its variations will be used also later on.) We represent $g \in \Pi_d$ from a small neighborhood of f as

$$g(z) = b(g) + \int_0^z g'(w)dw, \quad (9)$$

and this holds for every z in the plane. Hence, for a fixed k ,

$$\frac{\partial f}{\partial v_k}(z) = \frac{\partial b}{\partial v_k} + \int_0^z \frac{\partial g'}{\partial v_k}(w)dw. \quad (10)$$

On the other hand, for any j ,

$$v_j(g) = b(g) + \int_0^{c_j(g)} g'(w)dw. \quad (11)$$

As $c_j(g)$ is a holomorphic function of $V(g)$ and $g'(c_j(g)) = 0$, one can take the $\partial/\partial v_j$ derivative of (11) at the point $V(f)$ and write:

$$\delta_{j,k} = \frac{\partial b}{\partial v_k} + \frac{\partial c_j(g)}{\partial v_k} f'(c_j) + \int_0^{c_j} \frac{\partial g'}{\partial v_k}(w)dw = \frac{\partial b}{\partial v_k} + \int_0^{c_j} \frac{\partial g'}{\partial v_k}(w)dw. \quad (12)$$

Comparing it with (10), one gets:

$$\frac{\partial f}{\partial v_k}(z) - \delta_{j,k} = \int_{c_j}^z \frac{\partial g'}{\partial v_k}(w)dw. \quad (13)$$

As $c_j(g)$ is an m_j -multiple root of g' , we have:

$$\frac{\partial g'}{\partial v_k} = (z - c_j)^{m_j-1} r(z),$$

where $r(z)$ is a polynomial. Hence, as $z \rightarrow c_j$,

$$\frac{\partial f}{\partial v_k}(z) - \delta_{j,k} = (z - c_j)^{m_j} \frac{r(c_j)}{m_j} + O(z - c_j)^{m_j+1}. \quad (14)$$

□

Next statement is about an arbitrary rational function which fixes infinity.

Lemma 2.1 *Let f be any rational function so that $f(\infty) = \infty$. Let $c_j, j = 1, \dots, p$ be all geometrically different finite critical points of f , such that the corresponding critical values $v_j = f(c_j), j = 1, \dots, p$, are also finite. Denote by m_j the multiplicity of $c_j, j = 1, \dots, p$. Then there are functions $L_1(z), \dots, L_p(z)$ as follows. For every finite z , which is not a critical point of f , for every finite x , which is not a critical value of f and such that $x \neq f(z)$, we have:*

$$T \frac{1}{z-x} := \sum_{y: f(y)=x} \frac{1}{f'(y)^2} \frac{1}{z-y} = \frac{1}{f'(z)} \frac{1}{f(z)-x} + \sum_{j=1}^p \frac{L_j(z)}{x-v_j}. \quad (15)$$

Furthermore, each function L_j obeys the following two properties:

(1) L_j is a meromorphic function of the form:

$$L_j(z) = \sum_{i=1}^{m_j} \frac{q_{m_j-i}^{(j)}}{(z-c_j)^i}, \quad (16)$$

(2) for every $k = 1, \dots, p$, the function $f'(z)L_j(z) - \delta_{j,k}$ has zero at the point c_k of order at least m_j .

Proof. Fixing z, x as in the lemma, take R big enough and consider the integral $I = \frac{1}{2\pi i} \int_{|w|=R} \frac{dw}{f'(w)(f(w)-x)(w-z)}$. As the integrand is $O(1/w^2)$ at infinity, $I = 0$. On the other hand, applying the Residue Theorem,

$$I = -T \frac{1}{z-x} + \frac{1}{f'(z)} \frac{1}{f(z)-x} + \sum_{j=1}^p I_j(z, x).$$

Here

$$I_j(z, x) = \frac{1}{2\pi i} \int_{|w-c_j|=\epsilon} \frac{dw}{f'(w)(f(w)-x)(w-z)}.$$

Near $c = c_j$, $f'(w) = (w-c)^m r(w)$, where $m = m_j$ and $r = r_j$ is holomorphic with $r(c) \neq 0$. Denote

$$\frac{1}{r(w)} = \sum_{k=0}^{\infty} q_k (w-c)^k,$$

where $q_k = q_k^{(j)}$ and $q_0 = q_0^{(j)} = 1/r(c) \neq 0$. We can write:

$$\begin{aligned} \frac{1}{f'(w)(f(w)-x)(w-z)} &= \frac{1}{(w-c)^m r(w)((f(c)-x) + O((w-c)^{m+1}))(w-z)} = \\ &= \frac{1}{r(w)(f(c)-x)(w-z)} \frac{1}{(w-c)^m} + O(w-c) = \end{aligned}$$

$$\frac{1}{x-f(c)} \sum_{k=0}^{\infty} q_k(w-c)^{k-m} \sum_{n=0}^{\infty} \frac{(w-c)^n}{(z-c)^{n+1}} + O(w-c).$$

We see from here that $I_j(z, x) = L_j(z)/(x - f(c_j))$, where $L_j(z)$ has precisely the form (16). Now, consider $\tilde{L}(z) = f'(z)L_j(z)$. By (16), $\tilde{L}(z)$ has zero at every $c_k \neq c_j$ of order at least m_k . On the other hand, as $z \rightarrow c_j$, then

$$\begin{aligned} f'(z)L_j(z) &= r_j(z)[q_0^{(j)} + q_1^{(j)}(z - c_j) + \dots + q_{m_j-1}^{(j)}(z - c_j)^{m_j-1}] = \\ &= r_j(z)\left[\frac{1}{r_j(z)} - \sum_{k=m_j}^{\infty} q_k^{(j)}(z - c_j)^k\right] = 1 - (z - c_j)^{m_j}g(z), \end{aligned}$$

where g is holomorphic near c_j . This finishes the proof of the property (2). \square

As a simple corollary of the last two statements we have:

Proposition 3 *Let $f \in \Pi_{d,\bar{p}}$. Then*

$$T \frac{1}{z-x} = \frac{1}{f'(z)} \frac{1}{f(z)-x} + \sum_{k=1}^p \frac{\frac{\partial f}{\partial v_k}(z)}{f'(z)} \frac{1}{x-v_k}. \quad (17)$$

Proof. By the property (1), $f'(z)L_k(z)$ is a polynomial of degree at most $d-2$, which, by the property (2), coincides with the polynomial $p_k(z)$ introduced in Lemma 2. Therefore, indeed, $L_k(z) = \frac{\frac{\partial f}{\partial v_k}(z)}{f'(z)}$. \square

Proof of Part (a).

The following identity is easy to verify:

$$\frac{\partial f^m}{\partial v_k}(z) = (f^m)'(z) \sum_{n=0}^{m-1} \frac{\frac{\partial f}{\partial v_k}(f^n(z))}{(f^{n+1})'(z)}. \quad (18)$$

Letting here $z \rightarrow c_j$, one gets:

$$\frac{\partial f^m}{\partial v_k}(c_j) = (f^{m-1})'(v_j) \left\{ \frac{\partial f}{\partial v_k}(c_j) + \sum_{n=1}^{m-1} \frac{\frac{\partial f}{\partial v_k}(f^{n-1}(v_j))}{(f^n)'(v_j)} \right\}. \quad (19)$$

As we know, $\frac{\partial f}{\partial v_k}(c_j) = \delta_{j,k}$. Besides, $\frac{\partial f}{\partial v_k}(z) = f'(z)L_k(z) = p_k(z)$ is a polynomial of degree at most $d-2$. Hence, for some constant C_k and all z ,

$$\left| \frac{\partial f}{\partial v_k}(z) \right| \leq C_k(1 + |z|^{d-2}). \quad (20)$$

Now, assume that c_j is weakly expanding. As $c_j \in J$, the sequence $\{f^n(v_j)\}_{n \geq 0}$ is uniformly bounded. Then (20) and the summability condition imply that the series $\sum_{n=1}^{\infty} \frac{\frac{\partial f}{\partial v_k}(f^{n-1}(v_j))}{(f^n)'(v_j)}$ converges absolutely. Thus we have:

$$L(c_j, v_k) = \lim_{m \rightarrow \infty} \frac{\frac{\partial f^m}{\partial v_k}(c_j)}{(f^{m-1})'(v_j)} = \delta_{j,k} + \sum_{n=1}^{\infty} \frac{\frac{\partial f}{\partial v_k}(f^{n-1}(v_j))}{(f^n)'(v_j)}. \quad (21)$$

2.5 Proposition 1 and its corollary

To prove Proposition 1, we use (17) and write (in the formal series):

$$\begin{aligned} \varphi_{z,\lambda}(x) - \lambda(T\varphi_{z,\lambda})(x) &= \\ \sum_{n=0}^{\infty} \frac{\lambda^n}{(f^n)'(z)} \frac{1}{f^n(z) - x} - \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{(f^n)'(z)} \left\{ \frac{1}{f'(f^n(z))} \frac{1}{f^{n+1}(z) - x} + \sum_{k=1}^p \frac{\frac{\partial f}{\partial v_k}(f^n(v_j))}{f'(f^n(v_j))} \frac{1}{x - v_k} \right\} \\ &= \frac{1}{z - x} + \sum_{k=1}^p \frac{1}{v_k - x} \sum_{n=0}^{\infty} \lambda^{n+1} \frac{\frac{\partial f}{\partial v_k}(f^n(v_j))}{(f^{n+1})'(v_j)}. \end{aligned}$$

This ends the proof.

Putting in Proposition 1 $\lambda = 1$ and $z = v_j$ and combining it with (21), we get:

Proposition 4 *Let c_j be a summable critical point of $f \in \Pi_{d,\bar{p}}$. Then*

$$H_j(x) - (TH_j)(x) = \sum_{k=1}^p \frac{L(c_j, v_k)}{v_k - x}, \quad (22)$$

where

$$H_j(x) = \sum_{n=0}^{\infty} \frac{1}{(f^n)'(v_j)(f^n(v_j) - x)}, \quad (23)$$

2.6 Proof of Part (b) of Theorem 1

The proof is very similar to the one of Corollary 1(b) of [11], see also [14], [10] and references therein. Let c_j , $1 \leq j \leq r$, be weakly expanding. Assume the contrary, i.e., the rank of the matrix \mathbf{L} is less than r . Then, by (22), some non-trivial linear combination H of H_1, \dots, H_r is an integrable fixed point of T , which is holomorphic in each component of the complement $\mathbf{C} \setminus J$. Let us show that $H = 0$ off J . We use that T is weakly contracting. Consider a component Ω of $\mathbf{C} \setminus J$. If Ω is not a Siegel disk, then there is a domain U , such that $U \setminus f^{-1}(U)$ contains a non-empty open subset of Ω . We then have (the integration is against the Lebesgue measure on the

plane): $\int_U |H(x)| d\sigma_x = \int_U |TH(x)| d\sigma_x \leq \int_{f^{-1}(U)} |H(x)| d\sigma_x$, which is possible only if $H = 0$ in $U \setminus f^{-1}(U)$, hence, in Ω . And if Ω is a Siegel disk, we proceed as in the proof of Corollary 1(b) of [11] (p. 190) to show that $H = 0$ in Ω as well (see also Lemma 5.2). On the other hand, H has a form $H(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{x-b_k}$, where $b_k \in J$ and $|\alpha_k| < \infty$. Consider a measure with compact support $\mu = \sum_{k=0}^{\infty} \alpha_k \delta(b_k)$, where $\delta(z)$ is the Dirac measure at the point z . Then $H = 0$ off J means that the measure μ annihilates any function, which is holomorphic in a neighborhood of J . As every point of J belongs also to the boundary of the basin of infinity, by a corollary from Vitushkin's theorem (see e.g. [5]), every continuous function on J is uniformly approximated by rational functions. It follows, $\mu = 0$, a contradiction.

3 Rational functions

3.1 Local spaces of rational maps

Let f be a rational function of degree $d \geq 2$, such that

$$f(z) = \sigma z + m + \frac{P(z)}{Q(z)}, \quad (24)$$

where $\sigma \neq 0, \infty$, and Q, P are polynomials of degrees $d-1$ and at most $d-2$ respectively, which have no common roots. Without loss of generality, one can assume that $Q(z) = z^{d-1} + a_1 z^{d-2} + \dots + a_{d-1}$ and $P(z) = b_0 z^{d-2} + \dots + b_{d-2}$.

As in [10], we define a local (near f) space $\Lambda_{d, \bar{p}'}$ of rational functions g of the same degree d . Namely, let p' stand for the number of different critical points $c_1, \dots, c_{p'}$ of f . Denote by m_j the multiplicity of c_j , that is, the equation $f(w) = z$ has precisely $m_j + 1$ different solutions near c_j for z near $f(c_j)$ and $z \neq f(c_j)$, $j = 1, \dots, p'$. Observe that it is equivalent to say that, if we denote $f = \frac{\hat{P}}{\hat{Q}}$ (where $\hat{P}(z) = Q(z)(\sigma z + m) + P(z)$), then c_j , for $1 \leq j \leq p'$, is a root of the polynomial $\hat{P}'Q - \hat{P}Q'$ of multiplicity m_j . Note that $\sum_{j=1}^{p'} m_j = 2d - 2$.

Denote $\bar{p}' = \{m_j\}_{j=1}^{p'}$, the vector of multiplicities. Denote $v_j = f(c_j)$, $1 \leq j \leq p'$, the corresponding critical values. We assume that some of them can coincide as well as some can be ∞ . By p we denote the number of critical points of f with finite images, i.e. so that the corresponding critical values are finite. The space $\Lambda_{d, \bar{p}'}$ is defined as the set of all rational functions g of degree d of the same form

$$g(z) = \sigma(g)z + m(g) + \frac{P_g(z)}{Q_g(z)}, \quad (25)$$

where the numbers $\sigma(g), m(g)$, and the polynomials P_g, Q_g are close to σ, m, P, Q respectively. Moreover, g has p' different critical points $c_1(g), \dots, c_{p'}(g)$, so that $c_j(g)$ is close to c_j and has the same multiplicity m_j , $1 \leq j \leq p'$.

Two rational functions are called close if they are uniformly close in the Riemann metric on the sphere. We call two rational functions (M) -equivalent if there is a Mobius transformation M , which conjugates them. Every rational function f is equivalent to some \tilde{f} of degree $d \geq 2$ which belongs to some $\Lambda_{d,\bar{p}'}$. Indeed, f has either a repelling fixed point, or a fixed point with the multiplier 1 (see e.g. [13]). Hence, there exists a Mobius transformation P , such that ∞ is a fixed non-attracting point of $\tilde{f} = P \circ f \circ P^{-1}$. See also Section 3.2.

For g in a small enough neighborhood of f in $\Lambda_{d,\bar{p}'}$, introduce a vector $\bar{v}(g) \in \mathbf{C}^{p'+2}$ as follows. Let us fix an order $c_1, \dots, c_{p'}$ in the collection of all critical points of f . Moreover, we will do it in such a way, that first p indexes correspond to finite critical values, i. e. $v_j \neq \infty$ for $1 \leq j \leq p$ and $v_j = \infty$ for $p < j \leq p'$ (if $p < p'$). There exist p' functions $c_1(g), \dots, c_{p'}(g)$, which are defined and continuous in a small neighborhood of f in $\Lambda_{d,\bar{p}'}$, such that they constitute all different critical points of g of the multiplicities m_j . Define now the vector $\bar{v}(g)$. If all critical values of f are finite, then we set $\bar{v}(g) = \{\sigma(g), m(g), v_1(g), \dots, v_{p'}(g)\}$. If some of the critical values v_j of f are infinity, that is, $v_j = \infty$ for $p < j \leq p'$, then we replace in the definition of $\bar{v}(g)$ corresponding $v_j(g)$ by their reciprocals $v_j(g)^{-1}$:

$$\bar{v}(g) = \{\sigma(g), m(g), v_1(g), \dots, v_p(g), v_{p+1}(g)^{-1}, \dots, v_{p'}(g)^{-1}\}.$$

In particular, $\bar{v} = \bar{v}(f) = \{\sigma, m, v_1, \dots, v_p, 0, \dots, 0\}$. It is proved in [10], that $\Lambda_{d,\bar{p}'}$ is a complex-analytic manifold of dimension $p'+2$, and $\bar{v}(g)$ defines a local coordinate of $g \in \Lambda_{d,\bar{p}'}$. We remark that $p \geq 1$ (there always at least one finite critical value).

If a critical value of f or its iterate is infinity, we consider also another space of rational maps, which is biholomorphic to $\Lambda_{d,\bar{p}'}$. Let us fix a Mobius transformation M , such that $\alpha = M(\infty)$ lies outside of the critical orbits $\{f^n(c_k) : n \geq 0, 1 \leq k \leq p'\}$ of f . Then we make the same change of variable M for all maps from $\Lambda_{d,\bar{p}'}$: we get the space $\Lambda_{d,\bar{p}'}^M = \{M^{-1} \circ g \circ M : g \in \Lambda_{d,\bar{p}'}\}$, which is obviously biholomorphic to $\Lambda_{d,\bar{p}'}$. The advantage of the change is that the critical orbits of $\tilde{f} = M^{-1} \circ f \circ M$ lie in the plane (although can be unbounded). If $\tilde{g} \in \Lambda_{d,\bar{p}'}^M$, the points $c_k(\tilde{g}) = M^{-1}(c_k(g))$ and $v_k(\tilde{g}) = M^{-1}(v_k(g))$ are the critical point (of multiplicity m_k) and the critical value of \tilde{g} , $1 \leq k \leq p'$. Then the vector $\bar{v}^M(\tilde{g}) = \{\sigma(g), m(g), v_1(\tilde{g}), v_2(\tilde{g}), \dots, v_{p'}(\tilde{g})\}$ is a holomorphic coordinate system in $\Lambda_{d,\bar{p}'}^M$.

3.2 Subspaces

Suppose that f is an arbitrary rational function of degree $d \geq 2$. Denote by p' the number of different critical points of f in the Riemann sphere, and by \bar{p}' the vector of multiplicities at the critical points. As it was mentioned, there is an alternative: either **(H)** f has a fixed point a , such that $f'(a) \neq 0, 1$, or **(N)** the multiplier of every fixed point of f is either 0 or 1, and there is a fixed point with the multiplier 1. The case **(N)** is degenerate. We consider each case separately.

(H). Let P be a Mobius transformation, such that $P(a) = \infty$. Then $\tilde{f} = P \circ f \circ P^{-1}$ belongs to $\Lambda_{d,\bar{p}'}$. Moreover, P can be chosen uniquely in such a way, that $m(\tilde{f}) = 0$, and the critical value v_p of \tilde{f} is equal to 1. Let us define a submanifold $\Lambda_{\tilde{f}}$ of $\Lambda_{d,\bar{p}'}$ consisting of $g \in \Lambda_{d,\bar{p}'}$ in a neighborhood of \tilde{f} , such that $m(g) = 0$, and $v_p(g) = 1$. The coordinate $\bar{v}(g)$ in $\Lambda_{d,\bar{p}'}$ restricted to $\Lambda_{\tilde{f}}$ is obviously a coordinate in that subspace, which turns it in a p' -dimensional complex manifold.

(N). There are two sub-cases to distinguish.

(NN): f has a fixed point a , such that $f'(a) = 1$ and $f''(a) \neq 0$. Let P be a Mobius transformation, such that $P(a) = \infty$. Then $\tilde{f} = P \circ f \circ P^{-1}$ belongs to $\Lambda_{d,\bar{p}'}$. Moreover, P can be chosen uniquely in such a way, that $v_p(\tilde{f}) = 1$ and $m(\tilde{f}) = 1$. Then we define $\Lambda_{\tilde{f}}$ to be the set of all $g \in \Lambda_{d,\bar{p}'}$ in a neighborhood of \tilde{f} , such that $m(g) = 1$, and $v_p(g) = 1$. Coordinates in $\Lambda_{\tilde{f}}$ is defined as in the previous case, and it turns $\Lambda_{\tilde{f}}$ in a p' -dimensional complex manifold.

(ND): every fixed point with multiplier 1 is degenerate. Let a be one of them: $f'(a) = 1$ and $f''(a) = 0$. Then the Mobius map P can be chosen uniquely in such a way, that $\tilde{f}(z) = P \circ f \circ P^{-1}(z) = z + O(1/z)$, and \tilde{f} has a critical value equal to 1 in one attracting petal of ∞ , and equal to 0 in another attracting petal of ∞ . Then $\Lambda_{\tilde{f}}$ consists of $g \in \Lambda_{d,\bar{p}'}$ in a neighborhood of \tilde{f} , such that the critical value of g , which is close to $v_{p-1}(\tilde{f}) = 1$ is identically equal to 1, and the critical value of g , which is close to $v_p(\tilde{f}) = 0$, is identically equal to 0. Then $\Lambda_{\tilde{f}}$ is a p' -dimensional complex manifold.

It is easy to check that in any of these cases, every $\tilde{g} \in \Lambda_{d,\bar{p}'}$ is equivalent (by a linear conjugacy) to some $\tilde{g}_1 \in \Lambda_{\tilde{f}}$. Moreover, if we drop the condition that maps fix infinity and consider the set $X_{\tilde{f}}$ of *all* rational functions \hat{g} of degree d , which are close to \tilde{f} and such that \hat{g} has p' different critical points with the same corresponding multiplicities, then, since any such \hat{g} has a fixed point close to infinity, it is equivalent to some $\tilde{g} \in \Lambda_{d,\bar{p}'}$ and, hence, to some $\tilde{g}_1 \in \Lambda_{\tilde{f}}$. In any of the cases (H), (NN), (ND), we denote

$$\Lambda_f = \{g = P^{-1} \circ \tilde{g} \circ P : \tilde{g} \in \Lambda_{\tilde{f}}\},$$

where the Mobius map P is taken from the above. Thus we have

Proposition 5 *Every rational function f of degree $d \geq 2$ is equivalent to some $\tilde{f} \in \Lambda_{d,\bar{p}'}$, where \tilde{f} is of one and only one type: either H or NN or ND. If X_f denotes the set of all rational functions g of degree d , which are close to f and such that g have p' different critical points with the same corresponding multiplicities, then any $g \in X$ is equivalent to some $g_1 \in \Lambda_f$.*

3.3 Main result

Let us call a compact subset K of the Riemann sphere a **C-compact**, if there is a Mobius transformation M , such that $\hat{K} = M(K)$ is a compact subset of the

plane with the property that every continuous function on \hat{K} can be uniformly approximated by functions, which are holomorphic in a neighborhood of \hat{K} . Clearly, C-compact must have empty interior. Vitushkin's theorem characterizes such compacts on the plane, see e.g. [5]. As a simple corollary of this theorem, we have that each of the following conditions is sufficient for K to be a C-compact:

- (1) K has Lebesgue measure zero,
- (2) every $z \in K$ belongs to the boundary of a component of the complement of K .

We call a rational function f *exceptional* if f is *double covered by an integral torus endomorphism*: a family of explicitly described critically finite rational maps with Julia sets the Riemann sphere, see e.g. [1], [15].

Theorem 2 *Let $f \in \Lambda_{d, \bar{p}'}$. Fix a Mobius transformation M , such that $M(\infty)$ is disjoint with the forward orbits of the critical points of f . (If $f^n(c_j) \neq \infty$ for all $1 \leq j \leq p'$ and all $n \geq 0$, one can put M to be the identity map.) Consider $\tilde{f} = M^{-1} \circ f \circ M$ in the space $\Lambda_{d, \bar{p}'}^M$ with the coordinate \bar{v}^M . We denote by $c_k^M = M^{-1}(c_k)$, $v_k^M = M^{-1}(v_k)$, $1 \leq k \leq p'$, the corresponding critical points and critical values of \tilde{f} .*

(a) *Let c_j be a weakly expanding critical point of f . Then, for every $k = 1, \dots, p'$, the following limits exist:*

$$L^M(c_j, v_k) := \lim_{l \rightarrow \infty} \frac{\frac{\partial(\tilde{f}^l(c_j^M))}{\partial v_k^M}}{(\tilde{f}^{l-1})'(v_j^M)}, \quad (26)$$

$$L^M(c_j, \sigma) := \lim_{l \rightarrow \infty} \frac{\frac{\partial(\tilde{f}^l(c_j^M))}{\partial \sigma}}{(\tilde{f}^{l-1})'(v_j^M)}, \quad L^M(c_j, m) := \lim_{l \rightarrow \infty} \frac{\frac{\partial(\tilde{f}^l(c_j^M))}{\partial m}}{(\tilde{f}^{l-1})'(v_j^M)}. \quad (27)$$

Moreover, if $p < p'$, i.e., f has infinite critical values, then, for $p+1 \leq j \leq p'$ (i.e., when $v_j = \infty$) and $1 \leq k \leq p'$,

$$L^M(c_j, v_k) = \delta_{j,k}, \quad (28)$$

$$L^M(c_j, \sigma) = L^M(c_j, m) = 0. \quad (29)$$

(b) *Suppose that f is not exceptional, and f has r weakly expanding critical points. Without loss of generality, one can assume that c_{j_1}, \dots, c_{j_r} , $1 \leq j_1 < \dots < j_r \leq p'$, are such points, where $1 \leq j_1 < \dots < j_\nu \leq p$, the indexes of such points with finite images, and, if $\nu < r$, i.e., there are such critical points with corresponding critical values infinity, then $j_{\nu+1} = p' - (r - \nu - 1)$, $j_{\nu+2} = p' - (r - \nu - 2)$, ..., $j_r = p'$. Denote by K the union of the ω -limit sets of these critical points. Assume that K is a C-compact. Then the rank of the matrix \mathbf{L}^M defined below is equal to r .*

(H_∞). If f of the type **H**, i.e., $f(z) = \sigma z + O(1/z)$ as $z \rightarrow \infty$ and $v_p = 1$, then

$$\mathbf{L}^{\mathbf{M}} = (L^M(c_{j_i}, \sigma), L^M(c_{j_i}, v_1), \dots, L^M(c_{j_i}, v_{p-1}), L^M(c_{j_i}, v_{p+1}), L^M(c_{j_i}, v_{p'}))_{1 \leq i \leq r} \quad (30)$$

(NN_∞). If f of the type **NN**, i.e., $\sigma = 1$, $m \neq 0$, and $v_p = 1$, then

$$\mathbf{L}^{\mathbf{M}} = (L^M(c_{j_i}, v_1), \dots, L^M(c_{j_i}, v_{p-1}), L^M(c_{j_i}, v_{p+1}), \dots, L^M(c_{j_i}, v_{p'}))_{1 \leq i \leq r} \quad (31)$$

(ND_∞). Finally, if f of the type **ND**: $\sigma = 1$, $m = 0$, $v_{p-1} = 1$, $v_p = 0$, then

$$\mathbf{L}^{\mathbf{M}} = (L^M(c_{j_i}, v_1), \dots, L^M(c_{j_i}, v_{p-2}), L^M(c_{j_i}, v_{p+1}), \dots, L^M(c_{j_i}, v_{p'}))_{1 \leq i \leq r} \quad (32)$$

If we apply the part (b) of this Theorem exactly as we apply Theorem 1 in the Comment 1, and then apply Proposition 5, we get

Corollary 3.1 *Let f be an arbitrary rational function of degree $d \geq 2$, which is not an exceptional one. Suppose f has r summable critical points c_1, \dots, c_r , and the union of their ω -limit sets is a C -compact. Replacing if necessary f by its equivalent, one can assume that the forward orbits of c_1, \dots, c_r lie in the plane. Consider the set X_f of all rational functions of degree d , which are close enough to f and have the same number p' of different critical points with the same corresponding multiplicities. Then there is a p' -dimensional manifold Λ_f and its r -dimensional submanifold Λ , $f \in \Lambda_f \subset \Lambda \subset X$, with the following properties:*

- (a) *every $g \in X$ is equivalent to some $\hat{g} \in \Lambda_f$,*
- (b) *for every family $f_t \in \Lambda$ through f , such that $f_t(z) = f(z) + tu(z) + O(|t|^2)$ as $t \rightarrow 0$, if $u \neq 0$, then, for some $1 \leq j \leq r$,*

$$\lim_{m \rightarrow \infty} \frac{\frac{d}{dt}|_{t=0} f_t^m(c_j(t))}{(f^{m-1})'(v_j)} \neq 0. \quad (33)$$

Here $c_j(t)$ is the critical point of f_t , such that $c_j(0) = c_j$, and $v_j = f(c_j)$. Furthermore, if f and all the critical points of f are real, the above maps and spaces can be taken real.

4 Part (a) of Theorem 2

As in the polynomial case, we start by calculating partial derivatives of a function $g \in \Lambda_{d, \bar{p}'}$ w.r.t. the standard local coordinates of the space $\Lambda_{d, \bar{p}'}$, i.e., σ , m , and critical values, which are *not* infinity.

Proposition 6 *Let $f \in \Lambda_{d, \bar{p}'}$, and $f(z) = \sigma z + m + P(z)/Q(z)$.*

- (a) *For every finite critical value $v_k = f(c_k)$ of f , we have: $\frac{\partial f}{\partial v_k}(z)$ is a rational function $q_k(z)$ of degree $2d - 2$ and of the form $\frac{\tilde{P}(z)}{(Q(z))^2}$, where \tilde{P} is a polynomial of*

degree at most $2d - 3$, which is uniquely characterized by the following conditions: at $z = c_k$, the function $q_k(z) - 1$ has zero of order at least m_k ; at $z = c_j$, for every $j \neq k$, $1 \leq j \leq p$, $q_k(z)$ has zero of order at least m_j ; and, finally, at $z = c_j$, for every $p + 1 \leq j \leq p'$, the polynomial $\tilde{P}(z)$ has zero of order at least m_j . In particular, if c_k is simple (i.e., $m_k = 1$), then (8) holds.

(b) We have, as well:

$$\frac{\partial f}{\partial \sigma}(z) = \frac{z}{\sigma} f'(z), \quad \frac{\partial f}{\partial m}(z) = \frac{1}{\sigma} f'(z). \quad (34)$$

Proof. It is similar to the polynomial case. Here are details. If $g \in \Lambda_{d, \bar{p}'}$ is close to f , then $g(z) = \sigma(g)z + m(g) + \frac{P_g(z)}{Q_g(z)}$, where $P_g(z) = a_0(g)z^{d-2} + \dots$, $Q_g(z) = z^{d-1} + \dots$ are polynomials with the coefficients, which are holomorphic functions of the vector $\bar{v}(g)$, and $P_f = P$, $Q_f = Q$. Hence, $\frac{\partial f}{\partial v}(z)$ is indeed a rational function of degree $2d - 2$ and of the form $\frac{\tilde{P}(z)}{(Q(z))^2}$, where \tilde{P} is a polynomial of degree at most $2d - 3$. More precisely, if $f = \hat{P}/Q$, then

$$\tilde{P} = \frac{\partial \hat{P}}{\partial v_k} Q - \hat{P} \frac{\partial Q}{\partial v_k}. \quad (35)$$

It is thus enough to check that it satisfies the characteristic property of the rational function $q_k(z)$. We may write

$$g(z) = b(g) + \int_0^z g'(w) dw, \quad (36)$$

and this holds for every z in the plane with $g(z)$ is finite. Hence, for a fixed k ,

$$\frac{\partial f}{\partial v_k}(z) = \frac{\partial b}{\partial v_k} + \int_0^z \frac{\partial g'}{\partial v_k}(w) dw. \quad (37)$$

On the other hand, for any j so that v_j is finite,

$$v_j(g) = b(g) + \int_0^{c_j(g)} g'(w) dw. \quad (38)$$

Note that $c_j(g) = (2\pi i m_j)^{-1} \int_{|z - c_j(f)| = \epsilon} z g''(z) / g'(z) dz$ is a holomorphic function of $\bar{v}(g)$. Hence, one can take the $\partial / \partial v_j$ derivative of (38) and write:

$$\delta_{j,k} = \frac{\partial b}{\partial v_k} + \frac{\partial c_j(g)}{\partial v_k} f'(c_j) + \int_0^{c_j} \frac{\partial g'}{\partial v_k}(w) dw = \frac{\partial b}{\partial v_k} + \int_0^{c_j} \frac{\partial g'}{\partial v_k}(w) dw. \quad (39)$$

This allows us to proceed as follows:

$$\frac{\partial f}{\partial v_k}(z) - \delta_{j,k} = \int_{c_j}^z \frac{\partial g'}{\partial v_k}(w) dw. \quad (40)$$

As c_j is an m_j -multiple root of f' , we have:

$$\frac{\partial g'}{\partial v_k} = (z - c_j)^{m_j-1} r(z),$$

where $r(z)$ is a holomorphic function near c_j . Hence, as $z \rightarrow c_j$,

$$\frac{\partial f}{\partial v_k}(z) - \delta_{j,k} = (z - c_j)^{m_j} \frac{r(c_j)}{m} + O(z - c_j)^{m_j+1}. \quad (41)$$

Now, let $p+1 \leq j \leq p'$, i.e., c_j is a root of $Q(z)$, and $Q(z) = (z - c_j)^{m_j+1} \psi(z)$, where ψ is analytic near c_j and $\psi(c_j) \neq 0$. As c_j and the coefficients of ψ are analytic functions of v_k , Then $\frac{\partial Q}{\partial v_k} = (z - c_j)^{m_j} \tilde{\psi}(z)$, where $\tilde{\psi}(z)$ is analytic near c_j . From (35) above, we conclude that indeed \tilde{P} has root at c_j with multiplicity at least m_j . This proves (a). Let us prove (b). On the one hand,

$$\frac{\partial f}{\partial \sigma}(z) = z + \frac{R(z)}{(Q(z))^2} = \frac{\tilde{R}(z)}{(Q(z))^2},$$

where R is a polynomial of degree at most $2d-3$, and $\tilde{R} = zQ^2 + R$. In particular, $\frac{\partial f}{\partial \sigma}(z) = z + O(1/z)$. On the other hand, repeating the consideration from the case (a), we see that, for every $1 \leq j \leq p$, c_j is a m_j -multiple zero of $\frac{\partial f}{\partial \sigma}(z)$, and, for $p < j \leq p'$, c_j is a m_j -multiple zero of \tilde{R} . Therefore, $\frac{\partial f}{\partial \sigma}(z)/f'(z) = (z + O(1/z))/(\sigma + O(1/z^2))$ and is a rational function without poles, that is, it must be equal to z/σ . The proof for $\partial f/\partial m$ is very similar and is left to the reader.

□

As in the polynomial case, we then have:

Proposition 7 *Let $f \in \Lambda_{d,p'}$. Then*

$$T \frac{1}{z-x} = \frac{1}{f'(z)} \frac{1}{f(z)-x} + \sum_{k=1}^p \frac{\frac{\partial f}{\partial v_k}(z)}{f'(z)} \frac{1}{x-v_k}. \quad (42)$$

Comment 2 *Note that the sum is over the finite critical values of f only.*

Proof. We use Lemma 2.1 and the part (a) of Proposition 6. By the property (1), $f'(z)L_k(z)$ is a rational function of the form \tilde{P}/Q^2 , where \tilde{P} is a polynomial of degree at most $2d-3$, such that $f'(z)L_k(z) - 1$ has root at c_k with multiplicity m_k , and, for every $j \neq k$, $1 \leq j \leq p'$, the polynomial \tilde{P} has root at c_j with multiplicity m_j . Hence, $f'(z)L_k(z)$ coincides with the rational function $q_k(z)$. Therefore, indeed, $L_k(z) = \frac{\frac{\partial f}{\partial v_k}(z)}{f'(z)}$.

□

If $v_k = \infty$ or $z = \infty$, Proposition 6 is not useful. In that case, we replace the space $\Lambda_{d,\tilde{p}'}$ by the space $\Lambda_{d,\tilde{p}'}^M = \{M^{-1} \circ g \circ M : g \in \Lambda_{d,\tilde{p}'}\}$, where M is a non-linear Möbius transformation, i.e., $\alpha = M(\infty) \neq \infty$ and $\beta = M^{-1}(\infty) \neq \infty$, and such that the forward orbits of critical points of f are different from α . Then the forward critical orbits of $\tilde{f} = M^{-1} \circ f \circ M$ are finite, hence, for any $l > 0$, if g is close enough to f , then the first l iterates of the critical points of \tilde{g} are finite, too. A function $\tilde{g} = M^{-1} \circ g \circ M$ is a holomorphic function of $\sigma(g)$, $m(g)$, and the critical values $v_j^M(\tilde{g}) = M^{-1}(v_j(g))$ of \tilde{g} , $1 \leq j \leq p'$. By v_j^M, c_j^M we denote $v_j^M(\tilde{f})$, $c_j^M(\tilde{f})$, i.e., the critical values and points of \tilde{f} . In particular, $v_j^M = 0$ if and only if $p+1 \leq j \leq p'$. As usual, v_k denote the critical values of f , and $\partial \tilde{f}^n / \partial v_k^M$ means $\partial \tilde{g}^n / \partial v_k^M(\tilde{g})$ calculated at the point \tilde{f} .

Proposition 8 (a) Take $l \geq 1$. If $f^i(v_j) \neq \infty$ for $0 \leq i \leq l-1$, then

$$\frac{\frac{\partial \tilde{f}^l}{\partial v_k^M}(c_j^M)}{(\tilde{f}^{l-1})'(v_j^M)} = \frac{(M^{-1})'(v_j)}{(M^{-1})'(v_k)} \frac{\frac{\partial f^l}{\partial v_k}(c_j)}{(f^{l-1})'(v_j)}, \quad (43)$$

$$\frac{\frac{\partial \tilde{f}^l}{\partial \sigma}(c_j^M)}{(\tilde{f}^{l-1})'(v_j^M)} = (M^{-1})'(v_j) \frac{\frac{\partial f^l}{\partial \sigma}(c_j)}{(f^{l-1})'(v_j)}, \quad \frac{\frac{\partial \tilde{f}^l}{\partial m}(c_j^M)}{(\tilde{f}^{l-1})'(v_j^M)} = (M^{-1})'(v_j) \frac{\frac{\partial f^l}{\partial m}(c_j)}{(f^{l-1})'(v_j)} \quad (44)$$

(b) We have:

$$\frac{\partial \tilde{f}}{\partial v_k^M}(\beta) = 0, 1 \leq k \leq p', \quad \frac{\partial \tilde{f}}{\partial \sigma}(\beta) = \frac{\partial \tilde{f}}{\partial m}(\beta) = 0, \quad (45)$$

$$\frac{\partial \tilde{f}}{\partial v_k^M}(c_j^M) = \delta_{j,k}, \quad \frac{\partial \tilde{f}}{\partial \sigma}(c_j^M) = \frac{\partial \tilde{f}}{\partial m}(c_j^M) = 0, \quad 1 \leq k, j \leq p'. \quad (46)$$

Proof. (a) Since $\tilde{g}^l = M^{-1} \circ g^l \circ M$, $v_k(g) = M(v_k^M(\tilde{g}))$, where M is a fixed map, and by the conditions on v_k, c_j , the following calculations make sense:

$$\begin{aligned} \frac{\frac{\partial \tilde{f}^l}{\partial v_k^M}(c_j^M)}{(\tilde{f}^{l-1})'(v_j^M)} &= \frac{(M^{-1})'(f^l(M(c_j^M))) \frac{\partial f^l}{\partial v_k}(M(c_j^M)) \frac{\partial v_k(g)}{\partial v_k^M}}{(M^{-1})'(f^{l-1}(M(v_j^M))) (f^{l-1})'(M(v_j^M)) M'(v_j^M)} = \\ &= \frac{(M^{-1})'(f^{l-1}(v_j)) \frac{\partial f^l}{\partial v_k}(c_j) M'(v_k^M)}{(M^{-1})'(f^{l-1}(v_j)) (f^{l-1})'(v_j) M'(v_j^M)} = \frac{(M^{-1})'(v_j)}{(M^{-1})'(v_k)} \frac{\frac{\partial f^l}{\partial v_k}(c_j)}{(f^{l-1})'(v_j)}. \end{aligned}$$

The proof of (44) is similar.

(b) We repeat an argument from the proof of Proposition 6. Since $\tilde{g}(\beta) = \beta$ for every $\tilde{g} \in \Lambda_{d,\bar{p}'}^M$, where $\beta \neq \infty$ and fixed, we write:

$$\tilde{g}(z) = \beta + \int_{\beta}^z \tilde{g}'(w)dw. \quad (47)$$

Hence,

$$\frac{\partial \tilde{f}}{\partial v_k^M}(z) = \int_{\beta}^z \frac{\partial \tilde{g}'}{\partial v_k^M}(w)dw, \quad (48)$$

and similar for the derivatives w.r.t. σ and m . To show (45), it remains to put $z = \beta$.

As for the proof of (46), we get, exactly like we get (13):

$$\frac{\partial \tilde{f}}{\partial v_k^M}(z) - \delta_{j,k} = \int_{c_j^M}^z \frac{\partial \tilde{g}'}{\partial v_k^M}(w)dw, \quad (49)$$

and similar relations for the derivatives w.r.t. σ, m . It remains then to put $z = c_j^M$.

□

Proof of Part (a) of Theorem 2.

Here we state and prove a refined version of Theorem 2(a) introducing notations $L(c_j, v_k)$, $L(c_j, \sigma)$, $L(c_j, m)$ to be used later on. Recall that the notations $L^M(c_j, v_k)$, $L^M(c_j, \sigma)$, $L^M(c_j, m)$ are introduced in Theorem 2.

Theorem 3 *Let $f \in \Lambda_{d,\bar{p}'}$.*

(1) Assume v_k is finite, c_j is summable, and the orbit of c_j is finite (lies in the plane). Then the limits below exist and are expressed as the following series, which converge absolutely:

$$L(c_j, v_k) := \lim_{l \rightarrow \infty} \frac{\frac{\partial(f^l(c))}{\partial v_k}}{(f^{l-1})'(v)} = \delta_{j,k} + \sum_{n=1}^{\infty} \frac{\frac{\partial f}{\partial v_k}(f^{n-1}(v_j))}{(f^n)'(v_j)}, \quad (50)$$

$$L(c_j, \sigma) := \lim_{l \rightarrow \infty} \frac{\frac{\partial(f^l(c_j))}{\partial \sigma}}{(f^{l-1})'(v_j)} = \frac{1}{\sigma} \sum_{n=0}^{\infty} \frac{f^n(v_j)}{(f^n)'(v_j)}, \quad (51)$$

$$L(c_j, m) := \lim_{l \rightarrow \infty} \frac{\frac{\partial(f^l(c_j))}{\partial m}}{(f^{l-1})'(v_j)} = \frac{1}{\sigma} \sum_{n=0}^{\infty} \frac{1}{(f^n)'(v)}. \quad (52)$$

Furthermore,

$$L^M(c_j, v_k) = \frac{(M^{-1})'(v_j)}{(M^{-1})'(v_k)} L(c_j, v_k), \quad (53)$$

$$L^M(c_j, \sigma) = (M^{-1})'(v_j)L(c_j, \sigma), \quad L^M(c_j, m) = (M^{-1})'(v_j)L(c_j, m). \quad (54)$$

(2) Assume v_k is finite, c_j is summable and $f^l(v_j) = \infty$, for some minimal $l \geq 1$. Define in this case:

$$L(c_j, v_k) := \delta_{j,k} + \sum_{n=1}^l \frac{\frac{\partial f}{\partial v_k}(f^{n-1}(v_j))}{(f^n)'(v_j)}, \quad (55)$$

$$L(c_j, \sigma) := \frac{1}{\sigma} \sum_{n=0}^{l-1} \frac{f^n(v_j)}{(f^n)'(v_j)}, \quad L(c_j, m) := \frac{1}{\sigma} \sum_{n=0}^{l-1} \frac{1}{(f^n)'(v)}. \quad (56)$$

Then (53)-(54) hold.

(3) Assume $v_j = f(c_j) = \infty$. Then, for $1 \leq k \leq p'$,

$$L^M(c_j, v_k) = \delta_{j,k}, \quad (57)$$

$$L^M(c_j, \sigma) = L^M(c_j, m) = 0. \quad (58)$$

Proof. (1) It is enough to prove (50)-(52). Then, by (43)-(44), (53)-(54) follow. We can use the identity (19): for every $l > 0$,

$$\frac{\frac{\partial f^l}{\partial v_k}(c_j)}{(f^{l-1})'(v_j)} = \frac{\partial f}{\partial v_k}(c_j) + \sum_{n=1}^{l-1} \frac{\frac{\partial f}{\partial v_k}(f^{n-1}(v_j))}{(f^n)'(v_j)}. \quad (59)$$

As we know, $\frac{\partial f}{\partial v_k}(c_j) = \delta_{j,k}$. Besides, by the property (1) of L_j , $\frac{\partial f}{\partial v_k}(z) = f'(z)L_k(z) = q_k(z)$ is a rational function of the form \tilde{P}/Q^2 , where \tilde{P} is a polynomial, such that $f'(z)L_k(z)$ is finite at infinity. Hence, for some constant C_k and all z ,

$$|\frac{\partial f}{\partial v_k}(z)| \leq C_k(1 + |f(z)|^2). \quad (60)$$

Now, assume that c_j is weakly expanding. Then

$$\sum_{n=1}^{\infty} \frac{|\frac{\partial f}{\partial v_k}(f^{n-1}(v_j))|}{|(f^n)'(v_j)|} \leq C_k \frac{1 + |f^n(v_j)|^2}{|(f^n)'(v_j)|} < \infty. \quad (61)$$

The proof of existence of $L(c_j, \sigma)$ and $L(c_j, m)$ is similar to the proof for v_k -derivative. Indeed,

$$\frac{\partial f^l}{\partial \sigma}(z) = (f^l)'(z) \sum_{n=0}^{l-1} \frac{\frac{\partial f}{\partial \sigma}(f^n(z))}{(f^{n+1})'(z)}.$$

Letting here $z \rightarrow c_j$ and using the part (b) of Lemma 6, one gets:

$$\frac{\partial f^l}{\partial \sigma}(c_j) = (f^{l-1})'(v_j) \left\{ \frac{\partial f}{\partial \sigma}(c_j) + \sum_{n=1}^{l-1} \frac{\frac{\partial f}{\partial \sigma}(f^{n-1}(v_j))}{(f^n)'(v_j)} \right\} =$$

$$(f^{l-1})'(v_j) \frac{1}{\sigma} \{c_j f'(c_j) + \sum_{n=1}^{l-1} \frac{f^{n-1}(v_j) f'(f^{n-1}(v_j))}{(f^n)'(v_j)}\} = \frac{(f^{l-1})'(v_j)}{\sigma} \left\{ \sum_{n=1}^{l-1} \frac{f^{n-1}(v_j)}{(f^{n-1})'(v_j)} \right\},$$

and we get (51). Doing the same (with obvious changes) for the $\partial/\partial m$ -derivative, we get (52).

(2) We use (59) with \tilde{f} instead of f , and Proposition 8 (a)-(b). Note that $f^j(v_j) = \infty$ if and only if $\tilde{f}^j(v_j^M) = \beta$. Then $\tilde{f}^j(v_j^M) = \beta$ for every $j \geq l$, and, hence, for $j \geq l$,

$$\frac{\frac{\partial f^j}{\partial v_k^M}(c_j^M)}{(\tilde{f}^{j-1})'(v_j^M)} = \frac{\partial \tilde{f}}{\partial v_k^M}(c_j^M) + \sum_{n=1}^{j-1} \frac{\frac{\partial \tilde{f}}{\partial v_k^M}(\tilde{f}^{n-1}(v_j^M))}{(\tilde{f}^n)'(v_j^M)} = \delta_{j,k} + \sum_{n=1}^{l-1} \frac{\frac{\partial \tilde{f}}{\partial v_k^M}(\tilde{f}^{n-1}(v_j^M))}{(\tilde{f}^n)'(v_j^M)} = \frac{\frac{\partial f^l}{\partial v_k^M}(c_j^M)}{(\tilde{f}^{l-1})'(v_j^M)}, \quad (62)$$

while

$$\frac{\frac{\partial f^l}{\partial v_k^M}(c_j^M)}{(\tilde{f}^{l-1})'(v_j^M)} = \frac{(M^{-1})'(v_j)}{(M^{-1})'(v_k)} \frac{\frac{\partial f^l}{\partial v_k}(c_j)}{(f^{l-1})'(v_j)} = \frac{(M^{-1})'(v_j)}{(M^{-1})'(v_k)} L(c_j, v_k).$$

The proof of (53)-(54) is similar.

(3) follows directly from (62) and Proposition 8(b).

□

Corollary 4.1 *The rank of the matrix \mathbf{L}^M is bigger than or equal to $r' := (r - \nu) + r_0$, where $r_0 \leq \nu$ is the rank of the matrix \mathbf{L} , which is obtained from \mathbf{L}^M by the following two operations: firstly, we cross out the last $r - \nu$ lines (if $\nu < r$) and the last $p' - p$ rows (if $p < p'$), and secondly, we replace $L^M(c_j, v_k)$, $L^M(c_j, \sigma)$ by $L(c_j, v_k)$, $L(c_j, \sigma)$ respectively.*

In other words, in the case (H_∞) ,

$$\mathbf{L} = (L(c_j, \sigma), L(c_{j_i}, v_1), \dots, L(c_{j_i}, v_{p-1}))_{1 \leq i \leq \nu}, \quad (63)$$

In the case (NN_∞) ,

$$\mathbf{L} = (L(c_{j_i}, v_1), \dots, L(c_{j_i}, v_{p-1}))_{1 \leq i \leq \nu}, \quad (64)$$

and in the case (ND_∞) ,

$$\mathbf{L} = (L(c_{j_i}, v_1), \dots, L(c_{j_i}, v_{p-2}))_{1 \leq i \leq \nu}. \quad (65)$$

Proof. By Theorem 3 (3), on the i -line of \mathbf{L}^M , for $\nu < i \leq r$, all elements are 0, except for 1, which is on the intersection of i -line and i -row. This implies that the rank of \mathbf{L}^M is bigger than or equal to $r - \nu$ plus the rank of the matrix, which is left after the first operation. And the second operation preserves the rank, as it follows from Theorem 3 (1)-(2) (we use (53)-(54)).

□

4.1 Proposition 9 and its corollary

Since the proof of Proposition 1 is formal, we get:

Proposition 9 *Let $f \in \Lambda_{d,\bar{p}'}$. Given $z \in \mathbf{C}$, we define $l(z)$ to be the minimal l , such that $f^l(z) = \infty$. If there is no such l , then $l(z) = \infty$. We have:*

$$\varphi_{z,\lambda}(x) - \lambda(T\varphi_{z,\lambda})(x) = \frac{1}{z-x} + \lambda \sum_{k=1}^p \frac{1}{v_k-x} \Phi_k(\lambda, z), \quad (66)$$

where

$$\varphi_{z,\lambda}(x) = \sum_{n=0}^{l(z)-1} \frac{\lambda^n}{(f^n)'(z)} \frac{1}{f^n(z) - x}, \quad (67)$$

$$\Phi_k(\lambda, z) = \sum_{n=0}^{l(z)-1} \lambda^{n+1} \frac{\frac{\partial f}{\partial v_k}(f^n(z))}{(f^{n+1})'(z)}. \quad (68)$$

Putting in Proposition 9 $\lambda = 1$ and $z = v_j$ and combining it with the definition of $L(.,.)$ from Theorem 3, we get:

Proposition 10 *Let c_j , $1 \leq j \leq p$, be a summable critical point of $f \in \Lambda_{d,\bar{p}'}$, such that $v_j \neq \infty$. Then*

$$H_j(x) - (TH_j)(x) = \sum_{k=1}^p \frac{L(c_j, v_k)}{v_k - x}, \quad (69)$$

where

$$H_j(x) = \sum_{n=0}^{l(v_j)-1} \frac{1}{(f^n)'(v_j)(f^n(v_j) - x)}, \quad (70)$$

Comment 3 *Note that the sum in (69) is over the finite critical values. In particular, $l(v_j) \geq 1$ for $1 \leq j \leq p$.*

5 Part (b) of Theorem 2

5.1 The Wolff-Denjoy series

Let $\{\alpha_k\}_{k=0}^\infty, \{b_k\}_{k=0}^\infty$ be two sequences of complex numbers. Assume that

$$\sum_{k \geq 0} |\alpha_k| (1 + |b_k|^2) < \infty, \quad (71)$$

It implies that two series

$$A = \sum_{k \geq 0} \alpha_k, \quad B = \sum_{k \geq 0} \alpha_k b_k, \quad (72)$$

converge absolutely. Define

$$H(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{b_k - x}. \quad (73)$$

Then H is integrable in every disk $B_r = \{|x| < r\}$. Indeed, if σ_x denotes the element of the Lebesgue measure on the plane of the variable x , then

$$\int_{B_r} |H(x)| d\sigma_x \leq \sum_{k \geq 0} |\alpha_k| \int_{B_r} \frac{1}{|x - b_k|} d\sigma_x \leq 2\pi \sum_{k \geq 0} |\alpha_k| (r + |b_k|) < \infty.$$

We define a kind of regularization of H at infinity as

$$\hat{H}(x) = H(x) + \frac{A}{x} + \frac{B}{x^2} = \sum_{n \geq 0} \left(\frac{\alpha_n}{b_n - x} + \frac{\alpha_n}{x} + \frac{\alpha_n b_n}{x^2} \right).$$

The name is justified by the following claim (which must be known):

Lemma 5.1 *\hat{H} is integrable at infinity.*

Proof. For every n , the function $1/(b_n - x) + 1/x + b_n/x^2$ is integrable at infinity, and one can write, for $r > 0$:

$$\begin{aligned} \int_{|x| > r} \left| \frac{1}{b_n - x} + \frac{1}{x} + \frac{b_n}{x^2} \right| d\sigma_x &= |b_n| \int_{|x| > r/|b_n|} \left| \frac{1}{1 - x} + \frac{1}{x} + \frac{1}{x^2} \right| d\sigma_x \leq \\ |b_n| (C_1 + \int_{2 > |x| > r/|b_n|} \frac{1}{|x|^2} d\sigma_x) &\leq |b_n| (C_2 + 2\pi \ln \frac{|b_n|}{r}) \leq C_3 |b_n| (1 + |b_n|), \end{aligned}$$

where the constants C_i here and below depend only on r . Now, for every R big enough, and using the condition (71), we have:

$$\int_{r < |x| < R} |\hat{H}(x)| d\sigma_x \leq \sum_{k=0}^{\infty} |\alpha_k| \int_{|x| > r} \left| \frac{1}{b_k - x} + \frac{1}{x} + \frac{b_k}{x^2} \right| d\sigma_x \leq C_3 \sum_{k=0}^{\infty} |\alpha_k| |b_k| (1 + |b_k|) \leq C_4,$$

where C_4 does not depend on R .

□

We need a contraction property of the operator T . Suppose the function H is defined by (73) under the condition (71), and A, B are defined by (72). Denote by K the closure (on the Riemann sphere) of the set $\{b_k\}$. The proof of the following claim is a minor variation of [1], [4], [14], [10]

Lemma 5.2 *Assume that K has no interior points. Let f be a rational function with the asymptotics at infinity $f(z) = \sigma z + m + O(1/z)$, such that H is a fixed point of the operator T associated to f .*

(1) If $A = B = 0$, then either $H = 0$ on the complement K^c of K , or f is an exceptional map (double covered by an integral torus endomorphism).

(2) If either $|\sigma| \geq 1$ and $m = 0$, or $A = 0$ and $\sigma = 1$, then $H = 0$ on K^c , too.

Proof. Note that H is analytic in each component of K^c . Now, take R big enough and consider the disk $D(R) = \{|x| < R\}$. We claim that

$$\lim_{R \rightarrow \infty} \left\{ \int_{f^{-1}(D(R))} |H(x)| d\sigma_x - \int_{D(R)} |H(x)| d\sigma_x \right\} \leq 0. \quad (74)$$

Indeed, in the case (1), this follows at once from the integrability of H at infinity. In the case (2), the conditions on σ imply that there is $a > 0$, such that

$$f^{-1}(D(R)) \subset D(R + |m| + a/R) \quad (75)$$

(actually, $f^{-1}(D(R)) \subset D(R)$, if $|\sigma| > 1$). On the other hand,

$$\lim_{R \rightarrow \infty} \int_{R < |x| < R + |m| + a/R} |H(x)| d\sigma_x = 0. \quad (76)$$

Indeed, by Lemma 5.1, $H(x) = \hat{H}(x) - A/x - B/x^2$, where \hat{H} is integrable at infinity. In particular, $\lim_{R \rightarrow \infty} \int_{R < |x| < R + |m| + a/R} |\hat{H}(x)| d\sigma_x = 0$. But an easy calculation shows that the conditions in the case (2) guarantee that $\lim_{R \rightarrow \infty} \int_{R < |x| < R + |m| + a/R} \left| \frac{A}{x} + \frac{B}{x^2} \right| d\sigma_x = 0$, so (76) is proved. This, along with (75), gives us (74) in the case (2).

Let us show that

$$|TH(x)| = \sum_{w: f(w)=x} \frac{|H(w)|}{|f'(w)|^2} \quad (77)$$

almost everywhere. Indeed, otherwise there is a set $A \subset D(R_0)$ of positive measure (for some R_0) and $\delta > 0$, such that $|TH(x)| < (1 - \delta) \sum_{w: f(w)=x} \frac{|H(w)|}{|f'(w)|^2}$ on A . Then, for all $R > R_0$,

$$\begin{aligned} \int_{D(R)} |H(x)| d\sigma_x &= \int_{D(R) \setminus A} |H(x)| d\sigma_x + \int_A |H(x)| d\sigma_x = \int_{D(R) \setminus A} |TH(x)| d\sigma_x + \int_A |TH(x)| d\sigma_x < \\ &\int_{f^{-1}(D(R) \setminus A)} |H(x)| d\sigma_x + (1 - \delta) \int_{f^{-1}(A)} |H(x)| d\sigma_x = \int_{f^{-1}(D(R))} |H(x)| d\sigma_x - \delta \int_{f^{-1}(A)} |H(x)| d\sigma_x, \end{aligned}$$

which contradicts (74). With (77) holding almost everywhere, we proceed as in the above cited papers.

□

Lemma 5.3 *Let $\alpha_k, b_k, k \geq 0$ be two sequences of complex numbers, such that $\sum_{k \geq 0} |\alpha_k| < \infty$ and the closure K on the Riemann sphere of the set $\{b_k, k \geq 0\}$ is a C -compact K . If $H(x) = \sum_{k \geq 0} \alpha_k / (b_k - x)$ is equal to 0 outside of K , then $\alpha_k = 0$ for every k .*

Proof. (1) The case when K is a C -compact in the plane is classical, see e.g. [2]. (2) Now, assume that $\infty \in K$. There $x_0 \in \mathbf{C} \setminus K$. Let $\epsilon > 0$ be so that $|b_k - x_0| > \epsilon$ for every k . Let $c_k = b_k - x_0$. Then the function $H_1(y) = \sum_{k \geq 0} \alpha_k / (c_k - y)$ is equal to 0 outside of the compact $K_1 = K - x_0$, which is also a C -compact, but does not contain the origin. By the definition, the compact $K_2 = 1/K_1 = \{1/y : y \in K_1\}$ is a C -compact on the plane. Consider $H_2(z) = \sum_{k \geq 0} (\alpha_k / c_k) / (1/c_k - z)$. Since $|c_k| > \epsilon$, still $\sum_{k \geq 0} |\alpha_k / c_k| < \infty$. But, for every $z = 1/y$ outside of K_2 , so that y is outside of K_1 , $H_2(z) = -yH_1(y) = 0$. Then we apply the case (1). □

5.2 Proof of Theorem 2 (b)

By Corollary 4.1, it is enough to show that the rank of the matrix \mathbf{L} is equal to ν . Now, we follow closely the proof of Theorem 6 of [10]. Suppose first we are in the case \mathbf{H}_∞ . Assume the rank of the matrix \mathbf{L} is less than ν . Without loss of generality, one can assume that $j_1 = 1, \dots, j_\nu = \nu$. Then the vectors

$$(L(c_j, \sigma), L(c_j, v_1), L(c_j, v_2), \dots, L(c_j, v_{p-1})), 1 \leq j \leq \nu, \quad (78)$$

are linearly dependent. For every $1 \leq j \leq \nu$,

$$H_j(x) - (TH_j)(x) = \sum_{k=1}^p \frac{L(c_j, v_k)}{v_k - x}, \quad (79)$$

where

$$H_j(x) = \sum_{n=0}^{l(v_j)-1} \frac{1}{(f^n)'(v_j)(f^n(v_j) - x)}, \quad (80)$$

By the assumption, there exists a linear combination $H = \sum_{j=1}^\nu a_j H_j$, where not all a_j are zeroes, such that the following holds:

$$H(z) - (TH)(z) = \frac{L}{v_p - z}. \quad (81)$$

Here H has the form

$$H(z) = \sum_{k=0}^{\infty} \frac{\alpha_k}{b_k - z},$$

and since each $c_j, 1 \leq j \leq \nu$, is summable, the sequences α_k, b_k satisfy the condition (71). Recall the notations $A = \sum_{k \geq 0} \alpha_k$, $B = \sum_{k \geq 0} \alpha_k b_k$, and the regularization \hat{H} of H is:

$$\hat{H}(z) = H(z) + \frac{A}{z} + \frac{B}{z^2}.$$

We use the following asymptotics, as $z \rightarrow \infty$, which are easily checked (see e.g. the proof of Lemma 8.2 of [10]):

$$T \frac{1}{z} = \frac{1}{\sigma z} + \frac{m}{\sigma z^2} + O\left(\frac{1}{z^3}\right), \quad T \frac{1}{z^2} = \frac{1}{z^2} + O\left(\frac{1}{z^3}\right). \quad (82)$$

Now, we can rewrite (81) as follows:

$$\hat{H}(z) - (T\hat{H})(z) = \frac{A}{z} - A\left(\frac{1}{\sigma z} + \frac{m}{\sigma z^2}\right) - \frac{L}{z} - \frac{Lv_p}{z^2} + O\left(\frac{1}{z^3}\right),$$

or

$$\hat{H}(z) - (T\hat{H})(z) = \frac{A(1 - 1/\sigma) - L}{z} + \frac{-Am/\sigma - Lv_p}{z^2} + O\left(\frac{1}{z^3}\right).$$

But the function \hat{H} is integrable at infinity, hence, so is $\hat{H} - T\hat{H}$, and we can write:

$$I_R := \int_{|z| > R} |\hat{H}(z) - (T\hat{H})(z)| d\sigma_z \leq \int_{|z| > R} |\hat{H}(z)| d\sigma_z + \int_{f^{-1}(|z| > R)} |\hat{H}(z)| d\sigma_z.$$

It implies that $I_R \rightarrow 0$ as $R \rightarrow \infty$. Then, necessarily, one must hold:

$$A(1 - \frac{1}{\sigma}) - L = 0, \quad -\frac{Am}{\sigma} - Lv_p = 0. \quad (83)$$

In our case, $m = 0$, and since $v_p = 1 \neq 0$, then $L = 0$. In other words, H is a fixed point of T . Furthermore, $\sigma \neq 1$ and $L = 0$, then $A = 0$. Now we use that $\sum_{j=1}^{\nu} L(c_j, \sigma) = 0$. By (51), (56),

$$B = \sum_{k \geq 0} \alpha_k b_k = \sum_{j=1}^{\nu} a_j \sum_{n \geq 0} \frac{f^n(v_j)}{(f^n)'(v_j)} = \sigma \sum_{j=1}^{\nu} a_j L(c_j, \sigma) = 0. \quad (84)$$

Hence, the regularization \hat{H} of H takes the form:

$$\hat{H}(z) = H(z) + \frac{A}{z} + \frac{B}{z^2} = H(z),$$

i.e., H is an integrable (on the plane) fixed point of T . By Lemma 5.2, either $H(z) = 0$ for every z outside of the set K , or f is an exceptional rational function. The latter is excluded, hence, the former holds.

Remaining cases are quite similar.

(NN_∞). The relations (83) hold. Since $\sigma = 1$, then the first one gives us $L = 0$, and since $m \neq 0$, the second relation gives $A = 0$. Besides, (84) also holds. Then we end the proof as in the first case.

(ND_∞), i.e. $\sigma = 1$ and $m = 0$. Now, assuming the contrary, we get a non-trivial linear combination $H = \sum_{j=1}^\nu a_j H_j$, such that

$$H(z) - (TH)(z) = \frac{L_{p-1}}{v_{p-1} - z} + \frac{L_p}{v_p - z}. \quad (85)$$

Then

$$\hat{H}(z) - (T\hat{H})(z) = \frac{A}{z} - A\left(\frac{1}{\sigma z} + \frac{m}{\sigma z^2}\right) - \frac{L_{p-1} + L_p}{z} - \frac{L_{p-1}v_{p-1} + L_p v_p}{z^2} + O\left(\frac{1}{z^3}\right),$$

and since the function \hat{H} is integrable at infinity, one must hold:

$$A\left(1 - \frac{1}{\sigma}\right) - (L_{p-1} + L_p) = 0, \quad -\frac{Am}{\sigma} - (L_{p-1}v_{p-1} + L_p v_p) = 0. \quad (86)$$

But $\sigma = 1$, $m = 0$, and $v_{p-1} = 1 \neq 0 = v_p$, hence, $L_{p-1} = L_p = 0$. Thus H is a fixed point of T , and Lemma 5.2(2) ends the proof.

Comment 4 *Main results of the paper - Theorem 1 and Theorem 2 - can be extended as follows. Assume that, in addition to r summable critical points, the map f has r_a non-repelling periodic orbits (so that their multipliers are different from 1, and each superattracting cycle contains only a single and simple critical point). Let us extend the matrix \mathbf{L} in Theorem 1 or matrix $\mathbf{L}^{\mathbf{M}}$ in Theorem 2 by r_a lines as follows: on j -line, one writes the derivatives of the multiplier of the j th non-repelling periodic orbit w.r.t. the corresponding local coordinates. Then the rank of the extended matrix is maximal (equal to $r + r_a$). In the proof, one should add to the equations (79) similar equations (2) of [10] for the non-repelling orbits, and then proceed as in the proof of Theorems 1, 2 above, and Theorems 2, 6 of [10].*

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